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Homological Duality for Crossed Products

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0. INTRODUCTION

A group G is a duality group of dimension n if there exists a right G-module C and an element $e \in \operatorname{Tor}_n^{\mathbb{Z}G}(C, \mathbb{Z})$ such that the morphism induced by cap product namely

$$e \cap - : \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A) \xrightarrow{\sim} \operatorname{Tor}^{\mathbb{Z}G}_{n-i}(C \otimes_{\mathbb{Z}} A, \mathbb{Z})$$

is an isomorphism for each integer i and every left G-module A (see [BE]). Because of the duality of G, the cohomological dimension of G is n and in particular G is torsion free.

In addition to the above construction we also need to recall the construction of the crossed product $K_t^\alpha \Gamma$ (see [AR1]). In $K_t^\alpha \Gamma$, K is a commutative ring, Γ is a group acting on it via a homomorphism $t : \Gamma \to \operatorname{Aut}(K)$, and α is an element of the cohomology group $H^2(\Gamma, K^*)$ (K^* is the set of invertible elements of K viewed as a Γ-module). The crossed product $K_t^\alpha \Gamma$ is isomorphic, as a left K module to the direct sum $\coprod_{\sigma \in \Gamma} K u_\sigma$, while the product is defined by the rule

$$(x u_\sigma)(y u_\tau) = x \sigma(y) f(\sigma, \tau) u_{\sigma\tau} \qquad (x, y \in K, \sigma, \tau \in \Gamma),$$

where $f : \Gamma \times \Gamma \to K^*$ is a 2-cocycle representing α and $\sigma(y)$ is the $t(\sigma)$ action on y.

The crossed product $K_t^\alpha \Gamma$ is an associative K^Γ algebra (K^Γ is the subring of K fixed by Γ). Up to an isomorphism of algebras $K_t^\alpha \Gamma$ does not depend on the choice of the representative cocycle. In particular denote by $K_t \Gamma$ the crossed product in case α is trivial.

Using an analogous definition for the cap product over a trivial crossed product $K_t \Gamma$ (instead of over $\mathbb{Z}G$) we define naturally the notion of duality for crossed products.

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0.1. Definition. The crossed product $K_t\Gamma$ is said to possess homological duality (duality for short) of dimension n, if there exists a right $K_t\Gamma$-module C and an element $e \in \operatorname{Tor}_n^{K_t\Gamma}(C, K)$ such that the induced homomorphism by cap product

$$e \cap - : \operatorname{Ext}^i_{K_t\Gamma}(K, A) \to \operatorname{Tor}^{K_t\Gamma}_{n-i}(C \otimes_K A, K)$$

is an isomorphism for each integer i and every left $K_t\Gamma$ module A. Here K has a $K_t\Gamma$ structure given by

$$(xu_\sigma)\, y = x\sigma(y), \qquad xu_\sigma \in K_t\Gamma,\ y \in K,\ \sigma \in \Gamma.$$

We refer to K with this $K_t\Gamma$ structure as the $K_t\Gamma$ principal module.

We say that the crossed product $K_t\Gamma$ has Poincaré duality if there exists a homomorphism $\operatorname{sgn}: G \to \mathbb{Z}_2$ such that

(1) the dualizing module C is isomorphic to K, as a K module.

(2) with the identification of (1), the right action of $K_t\Gamma$ on C is defined by $y(xu_\sigma) = (\operatorname{sgn}(\sigma)\, \sigma^{-1}(xy))$.

We say that the Poincaré duality is orientable if the map sgn is trivial and non-orientable otherwise.

It is easy to show (by extension of scalars) that if Γ is a duality group (over $\mathbb{Z}$) of dimension n, then $K_t\Gamma$ is a crossed product with duality of the same dimension for any commutative ring K and any action of Γ on K. But we do more.

0.2. Theorem. *Let $K_t\Gamma$ be a crossed product where Γ contains a duality group G of finite index. If Γ is torsion free with regard to the ring K and the action t (see definition below) then $K_t\Gamma$ possesses duality as well.*

0.3. Definition. The group Γ is said to be torsion free with regard to the commutative ring K and the action $t: \Gamma \to \operatorname{Aut}(K)$ if for every finite subgroup S of Γ there is a unity decomposition; i.e., there is an element $x_S \in K$ s.t.

$$\sum_{\sigma \in S} \sigma(x_S) = 1.$$

Remark. To my knowledge this definition is new. It generalizes the classical one where the action t is trivial.

Using the duality property for a certain family of crossed products we are able to shed some light on the problem of the behaviour of the global dimension for some non-trivial crossed products $K_t^\alpha G$. Specifically, let K be a field. In [AR1] it has been proved that

$$\text{gl dim } K_t^\alpha G \leqslant \text{gl dim } K_t G \leqslant \text{gl dim } KG \tag{0.4}$$

and that if gl dim KG is finite then gl dim $K_t G =$ gl dim KG.

In [AR1], it is shown that the strict inequality gl dim $K^\alpha G <$ gl dim KG may occur in case gl dim $KG = \infty$. More difficult is to show that strict inequality may occur also in case that gl dim KG is finite. Shamsuddin in [Sh] has constructed a 2-cocycle for which gl dim $K^\alpha G = 1 <$ gl dim $KG = 2$ where G is the free abelian group on 2-generators. It turns out that Shamsuddin's result is the first and easy case of the following general result of Rosset [Ro]. If G is a free abelian group (of any rank) there exists an extension K of $\mathbb{C}$ and a cocycle α in $H^2(G, K^*)$ (G acting trivially on K) such that gl dim $K^\alpha G = 1$.

Noetherian crossed products are much discussed in McConnell's and Robson's book [MR]. Before we explain their result we recall their notation. If R is a ring and σ an automorphism of R then $R[x, x^{-1}, \sigma]$ denotes the "skew polynomial ring" defined like the usual Laurent polynomials ring but the multiplication satisfies

$$xr = \sigma(r)x \qquad \text{for every} \quad r \in R.$$

They show [MR, Theorem 7.9.11].

0.5. Theorem. *Let $S = R[x, x^{-1}, \sigma]$ be a skew polynomial ring where R is noetherian. Assume* gl dim $R = d$. *Then* gl dim $S = d + 1$ *if and only if there exists a right module M over S such that $pd_R M = d$ and M is finitely generated over R. Moreover, if N is an S module, then $pd_S N = d + 1$ if and only if N contains an S-submodule M with the above properties.*

If G is a poly-$\mathbb{Z}$ group of rank n and α is an element of $H^2(G, K^*)$ one can apply this theorem several times as follows. Let

$$\{1\} = G_1 \lhd G_1 \lhd \cdots \lhd G_n = G$$

be a composition series for G with all quotients $G_{i+1}/G_i \approx \mathbb{Z}$. Then

0.6. Corollary. *Let $K_t^\alpha G$ be a crossed product where G is poly-$\mathbb{Z}$ of rank n and K is a field. Then* gl dim $K_t^\alpha G = n$ *if and only if there exist $K_t^\alpha G_i$ modules M_i, $i = 1, 2, ..., n$ such that*

(1) *$M_{i-1} \subset M_i$ as $K_t^\alpha G_{i-1}$ modules*

(2) *M_i is finitely generated over $K_t^\alpha G_{i-1}$.*

(In $K_t^\alpha G_i$ the class α and the action t are restricted to G_i.)

Remark. This result can be easily extended to poly{cyclic or finite} groups provided that the global dimension of $K_t^\alpha G$ is finite.

Using noetherianity and the fact that each subgroup G_i (in the composition series for G) is normalized by G_{i+1}, it is not difficult to show that the existence of the $K_t^\alpha G_i$-modules M_i (in Corollary 0.6) implies the existence of a $K_t^\alpha G$-module M finitely generated over the field K. The converse implication is trivial. (We omit the direct proof of this fact since it will follow from a more general result.)

Combining Corollary 0.6 and the discussion above we have

0.7. COROLLARY. *Let $K_t^\alpha G$ be a crossed product where G is poly{cyclic or finite} of rank n and K is a field. Assume* $\operatorname{gl\,dim} K_t^\alpha G < \infty$. *Then* $\operatorname{gl\,dim} K_t^\alpha G = n$ *if and only if there exists a $K_t^\alpha G$-module M of finite dimension over K.*

In this paper we generalize this result to crossed products which belong to the following family.

0.8. *Notation.* A crossed product $K_t^\alpha G$ is in $\mathscr{A}$ if

(1) K is a field

(2) $K_t G$ is a crossed product with Poincaré Duality.

0.9. THEOREM (Proved in Section 3.2). *Let $K_t^\alpha G$ be a crossed product which belongs to the family $\mathscr{A}$. Assume* $\operatorname{gl\,dim} K_t G = n$. *Then for $K_t^\alpha G$-modules A and B (right and left, respectively)*

$$\operatorname{Tor}_n^{K_t^\alpha G}(A, B) \neq 0$$

if and only if A and B contain non-trivial $K_t^\alpha G$-submodules V_A and V_B, respectively, such that

$$\dim_K V_A = \dim_K V_B < \infty.$$

Moreover, if C denotes the dualizing module, then

$$C \otimes_K V_A \simeq \operatorname{Hom}_K(V_B, K) \qquad (= V_B^*) \tag{0.10}$$

as right $K_t^\alpha G$-modules. (The structures of the dual module V_B^* and the module $C \otimes_K V_A$ are explained in Section 1.)

0.11. COROLLARY. *Assume $K_t^\alpha G \in \mathscr{A}$. Then*

$$\textit{weak}\ \operatorname{gl\,dim} K_t^\alpha G = \operatorname{gl\,dim} K_t G$$

if and only if there exists a non-trivial $K_t^\alpha G$-module M of finite dimension over K. In particular if G is poly{cyclic or finite} such a module exists if and only if gl dim $K_t^\alpha G =$ gl dim $K_t G$.

Proof. If G is a poly{cyclic or finite} then the crossed product $K_t^\alpha G$ is a noetherian ring (see [MR, 1.5.11]). Now the result follows from the fact that for noetherian rings the weak global dimension and the global dimension (computed by the functor Tor and Ext, respectively) coincide (see [MR, 7.1.5, 7.1.8]).

In the proof of Theorem 0.9 we use cohomology of group methods. It seems that ring theory techniques cannot be applied to prove this theorem for the non-noetherian case, because the duality hypothesis which is the basis for our arguments is an essentially homological one.

At this point I thank the referee for his pointing out the relation between the above results and those appearing in [MR, Chap. 7]. To the author's belief, this mentioned relation can be applied to the subject dealt with in this paper.

At the end of Section 3.2 we prove another consequence of Theorem 0.9. Namely

0.12. PROPOSITION. *Assume $K_t G \in \mathscr{A}$. If* w.gl dim $K_t^\alpha G =$ gl dim $K_t G$ *then*:

(1) w.gl dim $K_t^{\alpha\beta} G =$ w.gl dim $K_t^\beta G$ *for all* $\beta \in H^2(G, K^*)$

(2) w.gl dim $K_t^{\alpha^{-1}} G =$ gl dim $K_t G$.

Recalling that w.gl dim $K_t G =$ gl dim $K_t G$ *for algebras in* $\mathscr{A}$, *we get that the subset of* $H^2(G, K^*)$

$$N = \{\alpha \in H^2(G, K^*) \colon \text{w.gl dim } K_t^\alpha G = \text{gl dim } K_t G\}$$

is a subgroup of $H^2(G, K^*)$ *and the* w.gl dim *function is well defined on the quotient* $H^2(G, K^*)/N$.

1. LEFT AND RIGHT STRUCTURES OF MODULES OVER CROSSED PRODUCT ALGEBRAS

1.1. If N is a left $K_t^\alpha G$-module we can furnish N with a right structure of $K_t^{\alpha^{-1}} G$ (α^{-1} is the inverse of α in the group $H^2(G, K^*)$) by defining

$$n(xw_\sigma) = v_\sigma^{-1} xn, \qquad n \in N,\ x \in K,\ w_\sigma \in K_t^{\alpha^{-1}} G,\ v_\sigma \in K_t^\alpha G.$$

To validate this definition, let us show that

$$(nxw_\sigma)\, yw_\tau = n(xw_\sigma\, yw_\tau).$$

The left hand side yields

$$\begin{aligned}(v_\sigma^{-1}xn)\, yw_\tau = v_\tau^{-1}yv_\sigma^{-1}xn &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(x)\, v_\tau^{-1}v_\sigma^{-1}n\\ &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(x)(v_\sigma v_\tau)^{-1}\, n\\ &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(x)(f(\sigma,\tau)\, v_{\sigma\tau})^{-1}\, n\\ &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(x)\, v_{\sigma\tau}^{-1}f^{-1}(\sigma,\tau)n\\ &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(xf^{-1}(\sigma,\tau))\, v_{\sigma\tau}^{-1}n\end{aligned}$$

while the right hand side yields

$$\begin{aligned}n(x\sigma(y)\, f^{-1}(\sigma,\tau)\, w_{\sigma\tau}) &= v_{\sigma\tau}^{-1}x\sigma(y)\, f^{-1}(\sigma,\tau)n\\ &= (\sigma\tau)^{-1}\,(x\sigma(y)\, f^{-1}(\sigma,\tau))\, v_{\sigma\tau}^{-1}n\\ &= \tau^{-1}(y)(\sigma\tau)^{-1}\,(xf^{-1}(\sigma,\tau))\, v_{\sigma\tau}^{-1}n.\end{aligned}$$

Note that we choose f^{-1} as representative for α^{-1} where f is the representative for α.

1.2. If M possesses a left $K_t^\alpha G$ structure and N possesses a left $K_t^\beta G$ structure then $M\otimes_K N$ can be given a left $K_t^{\alpha\beta}G$ structure by

$$yw_t(m\otimes n) = yu_\tau m\otimes v_\tau n$$

$$(n\in N,\ m\in M,\ y\in K,\ u_\tau\in K_t^\alpha G,\ v_\tau\in K_t^\beta G,\ w_\tau\in K_t^{\alpha\beta}G).$$

The tensor product $\otimes_K$ refers to the left structures of M and N over K (K is commutative). The proof that the action of $K_t^{\alpha\beta}G$ is well defined follows readily by choosing fg as a representative of $\alpha\beta$ where f and g are representative for α and β, respectively.

1.3. If M is a right $K_t^\alpha G$ module and N is a left $K_t^\beta G$ module then the diagonal action

$$w_\sigma(m\otimes n) = mu_\sigma^{-1}\otimes v_\sigma n, \qquad w_\sigma\in K_t^{\alpha^{-1}\beta}G,\ u_\sigma\in K_t^\alpha G,\ v_\sigma\in K_t^\beta G, \tag{1.3.1}$$

defines a left structure of $M\otimes_K N$ over $K_t^{\alpha^{-1}\beta}G$. This is easily understood if we remember that the right $K_t^\alpha G$ structure on M induces a left one over $K_t^{\alpha^{-1}}G$, which is the case dealt with in 1.1 (changing the sides). Now 1.3 follows using the case 1.2. In particular, if $\alpha=\beta$ then $M\otimes_K N$ has a K_tG structure; i.e., the cocycle disappears.

1.4. As in case 1.2 let N be a left $K_t^\alpha G$ module and N a left $K_t^\beta G$ module. In this case $\mathrm{Hom}_K(M, N)$ has a diagonal left structure over $K_t^{\alpha^{-1}\beta}G$ by

$$(xw_\sigma f)(m) = xv_\sigma f(u_\sigma^{-1}m), \qquad w_\sigma \in K_t^{\alpha^{-1}\beta}G,\ u_\sigma \in K_t^\alpha G,\ v_\sigma \in K_t^\beta G,\ x \in K. \tag{1.4.1}$$

This action is well defined since, if $m = zm'$, $z \in K$, z can be pulled out (the actions of v_σ, u_σ^{-1} cancel each other) establishing the K linearity of $xw_\sigma f$. By choosing the multiplication of representatives of α^{-1} and β to represent $\alpha^{-1}\beta$ one can verify that $xw_\sigma(yw_\tau f) = (xw_\sigma yw_\tau) f$.

Again we mention the case $\alpha = \beta$ in which $\mathrm{Hom}_K(M, N)$ is a left K_tG module (with no cocycle).

1.5. Finally a different right K_tG action on $\mathrm{Hom}_K(M, N)$ can be defined when a K_tG bi-module structure on N is provided (M is a left K module). This is done by

$$f(x\sigma)(m) = f(m)\, x\sigma.$$

2. Duality

2.1. *Duality Groups and Extensions of Scalars*

As mentioned in the Introduction the first step in the extension of duality for crossed products is to demonstrate that whenever G is a duality group of dimension n, the crossed product K_tG, has duality of the same dimension.

2.1.1. Proposition. *Let G be a duality group of dimension n and C the dualizing module, then for any unitary commutative ring K and any action $t\colon G \to \mathrm{Aut}(K)$, the crossed product K_tG has duality of the same dimension.*

Proof. The proof will be carried out by extending scalars. Let $F. \to \mathbb{Z} \to 0$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. For any left K_tG module A we have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}G}(F., A) \to \mathrm{Hom}_{K_tG}(K \otimes_{\mathbb{Z}} F., A)$$

$$u \to \bar{u}, \qquad \bar{u}(1 \otimes x) = u(x).$$

Here, $K \otimes_{\mathbb{Z}} F.$ possesses the K_tG structure defined by

$$x\sigma(y \otimes f) = x\sigma(y) \otimes \sigma f, \qquad \sigma \in G,\ x,\ y \in K,\ f \in F.$$

The resolution $F. \to \mathbb{Z} \to 0$ splits over $\mathbb{Z}$. Therefore $K \otimes_{\mathbb{Z}} F \to K \to 0$ is a $K_t G$ projective resolution of K. Moreover, it is easy to show that this isomorphism commutes with the respective differentials and thus we have isomorphisms of the cohomology groups

$$\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A) \simeq \mathrm{Ext}^*_{K_t G}(K, A).$$

Similarly a natural isomorphism of complexes exists in the tensor product

$$(C \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}G} F. \xrightarrow{\sim} ((K \otimes_{\mathbb{Z}} C) \otimes_K A) \otimes_{K_t G} (K \otimes_{\mathbb{Z}} F.)$$

which induces isomorphism of the homology groups

$$\mathrm{Tor}^{\mathbb{Z}G}_*(\dot{C} \otimes_{\mathbb{Z}} A, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Tor}^{K_t G}_*((K \otimes_{\mathbb{Z}} C) \otimes_K A, K).$$

Finally the natural homomorphism

$$C \otimes_{\mathbb{Z}G} F. \to (K \otimes_{\mathbb{Z}} C) \otimes_{K_t G} (K \otimes_{\mathbb{Z}} F.)$$

induces a homomorphism

$$\mathrm{Tor}^{\mathbb{Z}G}_n(C, \mathbb{Z}) \to \mathrm{Tor}^{K_t G}_n(K \otimes_{\mathbb{Z}} C, K).$$

Denote by e' the image of the fundamental class $e \in \mathrm{Tor}^{\mathbb{Z}G}_n(C, \mathbb{Z})$. We can deduce from naturality of the morphisms that

$$e' \cap - : \mathrm{Ext}^i_{K_t G}(K, A) \xrightarrow{\sim} \mathrm{Tor}^{K_t G}_{n-i}((K \otimes_{\mathbb{Z}} C) \otimes_K A, K)$$

for any left $K_t G$ module A. This means that the crossed product $K_t G$ has duality in homology of the same dimension as G and that the right $K_t G$ module $K \otimes_{\mathbb{Z}} C$ is the dualizing module. The action of $K_t G$ on $K \otimes_{\mathbb{Z}} C$ is defined by

$$(x \otimes c)\, y\sigma = \sigma^{-1}(xy) \otimes c\sigma, \qquad x, y \in K,\ c \in C,\ \sigma \in G.$$

This completes the proof.

2.2. Extension of Duality to Crossed Products $K_t G$ Where G Is a Virtual Duality Group

In the previous section we proved that whenever G is a duality group then the crossed product $K_t G$ (K commutative) has homological duality.

Now we wish to exhibit additional crossed products of the form $K_t G$ which have duality. But let us first state and analogous lemma to Shapiro's lemma, in the context of crossed products.

2.2.1. LEMMA (Shapiro). *Let Γ be a group acting on a commutative* *ring K (via a homomorphism t) and let H be a subgroup of Γ. If M is a $K_{t_0}H$ module (t_0 is the restriction of t on H) then there exist isomorphisms in cohomology and homology*

$$\operatorname{Ext}^r_{K_{t_0}H}(K, M) \simeq \operatorname{Ext}^r_{K_t\Gamma}(K, \operatorname{Hom}_{K_{t_0}H}(K_t\Gamma, M))$$
$$\operatorname{Tor}_r^{K_{t_0}H}(M, K) \simeq \operatorname{Tor}_r^{K_t\Gamma}(K_t\Gamma \otimes_{K_{t_0}H} M, K).$$

The proof is similar to the classical lemma.

2.2.3. THEOREM. *Let Γ be a virtual duality group of virtual dimension n; i.e., it contains an n-dimensional duality group H of finite index. Assume an action of Γ on a commutative ring K. If Γ has no torsion in K with the action t (see Definition 0.3) then the crossed product $K_t\Gamma$ has homological duality of the same dimension.*

Proof. As in [BE, Theorem 3.3] we need sufficient conditions for duality of a crossed product [BE, Sect. 2]. Indeed such conditions exist, and because of the great similarity of the statements and proofs in [BE] to ours we state the conditions without proof of their sufficiency.

2.2.2. DEFINITION. A K_tG-module A is said to be induced if it is isomorphic to $L \otimes_k kG$ over K_tG where k is the fixed ring K^G and L is a K_tG module. The action of K_tG on $L \otimes_k kG$ is defined by

$$xu_\sigma(l \otimes \tau) = xu_\sigma l \otimes \sigma\tau.$$

Note that this generalizes the classical definition over $\mathbb{Z}G$ (see Remark in [Se, V11, Sect. 1]).

2.2.4. PROPOSITION. *The following conditions are (necessary and) sufficient for a crossed product $K_t\Gamma$ to have duality of dimension $n \geqslant 0$*

(1) $\operatorname{proj\,dim}_{K_t\Gamma} K < \infty$ *(in fact equal to n).*

(2) $\operatorname{Ext}^r_{K_t\Gamma}(K, A) = 0$ *for $r \neq n$ and all induced modules A.*

(3) *There is an element $e \in \operatorname{Tor}_n^{K_t\Gamma}(C, K)$ ($C = \operatorname{Ext}^n_{K_t\Gamma}(K, K_t\Gamma)$) such that ($e \cap -$) induces an isomorphism*

$$\operatorname{Ext}^n_{K_t\Gamma}(K, F) \simeq C \otimes_{K_t\Gamma} F$$

for all free $K_t\Gamma$ modules F.

We will show that the crossed product $K_t\Gamma$ satisfies these conditions. Assuming condition (1) is fulfilled, let us first show that conditions (2) and (3) are satisfied.

Let N be the subgroup

$$N = \bigcap_{\sigma_i \in \Gamma} \sigma_i H \sigma_i^{-1},$$

clearly N is a normal subgroup in Γ of finite index. By [BE, Theorem 3.8] (since $N \leqslant H$ of finite index) N is a duality group of dimension n. Thus we can assume that H is a normal subgroup of Γ.

Let A be an induced $K_t\Gamma$ module. Clearly, A is an induced $\mathbb{Z}\Gamma$ module and in particular an induced $\mathbb{Z}H$ module. H is a duality group (over $\mathbb{Z}$) of dimension n. Therefore

$$H^r(H, A) = \operatorname{Ext}^r_{\mathbb{Z}H}(\mathbb{Z}, A) = 0 \qquad \text{for each} \quad r \neq n.$$

Recall the Hochschild–Serre spectral sequence for the short exact sequence

$$1 \to H \to \Gamma \to G \to 1.$$

The E_2 elements are given by

$$E_2^{p,q} = H^p(\Gamma/H, H^q(H, A))$$

and in the limit $E_2^{p,q} \Rightarrow E_\infty^{p,q}$ where E_∞ is the graded group $H^*(\Gamma, A)$, appropriately filtered. $E_2^{p,q}$ are the elements of degree $p+q$. Since $H^q(H, A) = 0$ for $q < n$, $H^r(\Gamma, A) = 0$ for $r < n$. Using the isomorphism $\operatorname{Ext}^r_{K_t\Gamma}(K, A) \simeq \operatorname{Ext}^t_{\mathbb{Z}\Gamma}(\mathbb{Z}, A)$ and recalling condition (1) ($\operatorname{proj\,dim}_{K_t\Gamma} K \leqslant n$), we proved condition (2).

To prove condition (3) note first that by the Shapiro's isomorphism (Lemma 2.2.1) $K_t\Gamma$ should have the same dualizing module as $K_{t_0}H$ where t_0 is the restriction of the action t on H. Namely

$$C = \operatorname{Ext}^n_{K_t\Gamma}(K, K_t\Gamma) \simeq \operatorname{Ext}^n_{K_{t_0}H}(K, K_{t_0}H). \tag{2.2.5}$$

We need to show the existence of an element $\bar{e} \in \operatorname{Tor}_n^{K_t\Gamma}(C, K)$, such that the map induced by the cap product with $\bar{e}$

$$\bar{e} \cap - : \operatorname{Ext}^n_{K_t\Gamma}(K, F) \simeq C \otimes_{K_t\Gamma} F$$

is an isomorphism for all free $K_t\Gamma$ modules F. Indeed, let $\hat{C}$ be the dualizing module of the duality group H. Then by Proposition 2.1.1, $K \otimes_{\mathbb{Z}} \hat{C}$ is the dualizing module for the crossed product $K_{t_0}H$ and by isomorphism 2.2.5 this is the candidate to be the dualizing module for $K_t\Gamma$. Now, by [BE, Theorem 3.3] there exists an element $e \in \operatorname{Tor}_n^{\mathbb{Z}\Gamma}(\hat{C}, \mathbb{Z})$ s.t. the map

$$e \cap - : \operatorname{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \simeq \hat{C} \otimes_{\mathbb{Z}\Gamma} F$$

is an isomorphism. Thus we get

$$\mathrm{Ext}^n_{K_t\Gamma}(K, F) \simeq \mathrm{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \overset{e\cap -}{\simeq} \hat{C} \otimes_{\mathbb{Z}\Gamma} F \simeq (K \otimes_{\mathbb{Z}} \hat{C}) \otimes_{K_t\Gamma} F.$$

It is not difficult to show that the compound isomorphism $(\mathrm{Ext}^n_{K_t\Gamma}(K, F) \simeq (K \otimes_{\mathbb{Z}} \hat{C}) \otimes_{K_t\Gamma} F)$ is the induced map by the cap product with the image of e, $\bar{e}$

$$\mathrm{Tor}^{\mathbb{Z}\Gamma}_n(\hat{C}, \mathbb{Z}) \to \mathrm{Tor}^{K_t\Gamma}_n(K \otimes_{\mathbb{Z}} \hat{C}, K) = \mathrm{Tor}^{K_t\Gamma}_n(C, K)$$

$$e \mapsto \bar{e}.$$

Hence condition (3).

We now turn to prove that condition (1) is satisfied.

2.2.6. Proposition. *Let Γ be a group acting on a commutative ring K. $K_t\Gamma$ is the associated crossed product. If Γ contains a subgroup H of finite index with $cd_{K_{t_0}}(H) \equiv \mathrm{proj\,dim}_{K_{t_0}H} K < \infty$ (t_0 is the restriction of t on H) and Γ has no torsion in K with the action t (see Definition 0.3) then $cd_{K_t}(\Gamma) < \infty$.*

Remark. This is a generalization of Serre's theorem to trivial crossed products $K_t\Gamma$. It seems to me that this is a natural context for Serre's theorem. Our proof is based on the classical proof in [Sw, Pa]. The difference lies in the fact that the K action does not commute with the Γ action. I believe that a partial proof with many references to the proof of the old theorem would have been almost impossible to follow. Therefore I give here a complete proof.

Proof. If P is a $K_{t_0}H$ module, there exists a decomposition over K

$$K_t\Gamma \otimes_{K_{t_0}H} P = \sum_i \sigma_i P,$$

where $\{\sigma_i\}_{i=1}^n$ are representatives of the cosets of H in Γ, i.e., $\Gamma = \bigcup_{i=1}^n \sigma_i H$.

Let $P. \to K \to 0$ be a finite dimensional projective resolution of K over $K_{t_0}H$. Since $P.$ and K are projective K-modules this resolution splits over K. Therefore the tensor product

$$Q = \bigotimes_{i=1}^n P_i = P_1 \otimes_K P_2 \otimes \cdots \otimes_K P_n \qquad (P_i \simeq \sigma_i P \text{ over } K)$$

is a resolution of K over K. The tensor product Q satisfies the condition

$$p_1 \otimes_K \cdots \otimes x p_i \otimes \cdots \otimes p_m = \sigma_i(x)(p_1 \otimes \cdots \otimes p_i \otimes \cdots \otimes p_n).$$

Our purpose is to furnish this complex with a $K_t\Gamma$ action so that Q becomes a $K_t\Gamma$ projective module and so that this action commutes with the respective differentials. Once this is proven the finiteness of the complex Q completes the proof.

Definition of the $K_t\Gamma$ Action

Every $g \in \Gamma$ defines $\sigma_{\nu_i} \in \{\sigma_j\}$ and $h_{\nu_i} \in H$ by

$$g^{-1}\sigma_i = \sigma_{\nu_i} h_{\nu_i}^{-1}, \qquad \nu \in \mathrm{Sym}(n).$$

Define

$$g(p_1 \otimes \cdots \otimes p_n) = (-1)^{r(g, p_1, ..., p_n)} h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n},$$

where $r(g, p_1, ..., p_n) = \sum \deg p_i \deg p_j$ and the sum is taken over the ordered pairs (i, j) in which $i < j$ but $\nu_i > \nu_j$, i.e., over those ordered pairs which "cross" each other during the permutation on the cosets induced by g. To show that this is a well defined action we prove

(1) $g(p_1 \otimes \cdots \otimes xp_i \otimes \cdots \otimes p_n) = g(\sigma_i(x)(p_1 \otimes \cdots \otimes p_n))$

(2) $(g_2 g_1)(p_1 \otimes \cdots \otimes p_n) = g_2(g_1(p_1 \otimes \cdots \otimes p_n))$

(3) this action commutes with the respective differentials.

As before for all $\sigma_i \in \{\sigma_j\}$ let $g^{-1}\sigma_i = \sigma_{\nu_i} h_{\nu_i}^{-1}$ and assume $\nu_j = i$. (This is abuse of notation since $j = j(i)$ but should not confuse the reader.)

Proof of (1). The left hand side in (1) yields

$$\begin{aligned} &g(p_1 \otimes \cdots \otimes xp_i \otimes \cdots \otimes p_n) \\ &\quad = (-1)^{r(g, p_1, ..., p_n)} h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_j} x p_{\nu_j} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n} \\ &\quad = (-1)^{r(g, p_1, ..., p_n)} \sigma_j h_{\nu_j}(x)(h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n}) \end{aligned}$$

while the right hand side of (1) yields

$$\begin{aligned} &g(\sigma_i(x)(p_1 \otimes \cdots \otimes p_i \otimes \cdots \otimes p_n)) \\ &\quad = g\sigma_i(x)(-1)^{r(g, p_1, ..., p_n)} h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n}. \end{aligned}$$

Since $\nu_j = i$, then $g^{-1}\sigma_j = \sigma_{\nu_j} h_{\nu_j}^{-1} = \sigma_i h_i^{-1}$ or $g\sigma_i = \sigma_j h_{\nu_j}$ showing (1).

Proof of (2). If $g_1^{-1}\sigma_i = \sigma_{\nu_i} h_{\nu_i}^{-1}$, $g_2^{-1}\sigma_i = \sigma_{\tau_i} \hat{h}_{\tau_i}^{-1}$ for each i then we have

$$\begin{aligned} (g_2 g_1)^{-1} \sigma_i &= g_1^{-1}(g_2^{-1}\sigma_i) = g_1^{-1}(\sigma_{\tau_i} \hat{h}_{\tau_i}^{-1}) = (g_1^{-1}\sigma_{\tau_i}) \hat{h}_{\tau_i}^{-1} \\ &= \sigma_{\nu_{\tau_i}} h_{\nu_{\tau_i}}^{-1} \hat{h}_{\tau_i}^{-1} = \sigma_{\nu_{\tau_i}} (\hat{h}_{\tau_i} h_{\nu_{\tau_i}})^{-1}. \end{aligned}$$

Now, the right hand side of (2) yields

$$
\begin{aligned}
&g_2(g_1(p_1 \otimes \cdots \otimes p_n)) \\
&\quad = g_2(-1)^{r(g_1, p_1, \ldots, p_n)} h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n} \\
&\quad = (-1)^{r(g_2, p_{\nu_1}, \ldots, p_{\nu_n})} (-1)^{r(g_1, p_1, \ldots, p_n)} \hat{h}_{\tau_1} h_{\nu_{\tau_1}} p_{\nu_{\tau_1}} \otimes \cdots \otimes \hat{h}_{\tau_n} h_{\nu_{\tau_n}} p_{\nu_{\tau_n}},
\end{aligned}
$$

while the left hand side gives

$$(g_2 g_1)(p_1 \otimes \cdots \otimes p_n) = (-1)^{(g_2 g_1, p_1, \ldots, p_n)} \hat{h}_{\tau_1} h_{\nu_{\tau_1}} p_{\nu_{\tau_1}} \otimes \cdots \otimes \hat{h}_{\tau_n} h_{\nu_{\tau_n}} p_{\nu_{\tau_n}}.$$

Then (2) will follow if we show

$$(-1)^{r(g_2, p_{\nu_1} \cdots p_{\nu_n})} (-1)^{r(g_1, p_1, \ldots, p_n)} = (-1)^{r(g_2 g_1, p_1, \ldots, p_n)}.$$

To see this, note that the permutation action is a group action; i.e., if p_i and p_j "cross" each other twice by the g_1 and g_2 actions, then they will not cross each other by the $g_2 g_1$ action. In other words the contribution of $\deg p_i \deg p_j$ is the same on both sides, proving (2).

Proof of (3). If $g \in \Gamma$ and d denotes the respective differentials we need to show that $dg(p_1 \otimes \cdots \otimes p_n) = gd(p_1 \otimes \cdots \otimes p_n)$.

As before, let $g^{-1}\sigma_i = \sigma_{\nu_i} h_{\nu_i}^{-1}$.

The left hand side gives

$$
\begin{aligned}
dg(p_1 \otimes \cdots \otimes p_n) &= d(-1)^{r(g, p_1, \ldots, p_n)} h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n} \\
&= (-1)^{r(g, p_1, \ldots, p_n)} \sum_{i=1}^{n} (-1)^{\deg p_{\nu_0} + \cdots + \deg p_{\nu_{i-1}}} h_{\nu_1} p_{\nu_1} \\
&\quad \otimes \cdots \otimes dh_{\nu_i} p_{\nu_i} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n}, \qquad (2.2.7)
\end{aligned}
$$

where by definition $\deg p_{\nu_0} = 0$.

While the right hand side gives

$$
\begin{aligned}
&gd(p_1 \otimes \cdots \otimes p_n) \\
&\quad = g\left(\sum_{i=1}^{n} (-1)^{\deg p_0 + \cdots + \deg p_{i-1}} p_1 \otimes \cdots \otimes dp_i \otimes \cdots \otimes p_n \right)
\end{aligned}
$$

and assuming $\nu_j = i$ (i.e., i passes to the jth place) we have

$$
\begin{aligned}
&= \sum_{i=1}^{n} (-1)^{\deg p_0 + \cdots + \deg p_{i-1}} (-1)^{r(g, p_1, \ldots, p_{i-1}, dp_i, p_{i+1} \cdots p_n)} \\
&\quad \times h_{\nu_1} p_{\nu_1} \otimes \cdots \otimes h_{\nu_j} dp_{\nu_j} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n}. \qquad (2.2.8)
\end{aligned}
$$

Consider the elements in sums 2.2.7 and 2.2.8 in which $v_i = s$. Since the differentials d commute with the $K_{t_0}H$ action we have to show

$$\begin{aligned}(-1)^{r(g, p_1 \cdots p_n)}\,(-1)^{\deg p_{v_0} + \cdots + \deg p_{v_{i-1}}} \\ = (-1)^{\deg p_0 + \cdots + \deg p_{s-1}}\,(-1)^{r(g, p_1 \cdots dp_s \cdots p_n)}.\end{aligned} \tag{2.2.9}$$

To see this, a little effort is required. Let $p_{s+t_1}, p_{s+t_2}, ..., p_{s+t_k}$ be the p_i's that cross p_s from its right to its left as a result of the permutation induced by g and similarly let $p_{s-l_1}, p_{s-l_2}, ..., p_{s-l_r}$ be the p_i's that cross p_s in the other direction. $v_i = s$ means that the element which was in the sth place before the permutation is moved to the ith place by the permutation. Since we have k new elements to the left of P_s while r other elements leave this side, we conclude that

$$s + k - r = i.$$

Since $\deg dp_s = \deg p_s - 1$ the signs $(-1)^{r(g, p_1, ..., p_n)}$ and $(-1)^{r(g, p_1, ..., dp_s, ..., p_n)}$ differ by the sign $(-1)^{\deg p_{s+t_1} + \cdots + \deg p_{s+t_k}}\,(-1)^{\deg p_{s-l_1} + \cdots + \deg p_{s-l_r}}$. Also the signs $(-1)^{\deg p_{v_0} + \cdots + p_{v_{i-1}}}$ and $(-1)^{\deg p_0 + \cdots + p_{s-1}}$ differ by exactly the same sign. This proves 2.2.9 and hence (3).

So far we have proved that Q is a resolution of the $K_t\Gamma$ module K. It remains to show that Q is a projective resolution. Since the module Q is the direct sum of $K_t\Gamma$ submodules, Q_d

$$Q_d = \sum_{i_1 + i_2 \cdots + i_n = d} P_{1, i_1} \otimes \cdots \otimes P_{n, i_n}$$

we can disregard the structure of a cochain complex and show that Q itself is a $K_t\Gamma$ projective module.

Consider P as a $K_{t_0}H$ projective module. Let P' be a $K_{t_0}H$ projective module such that $P \otimes P' = F$ is free over $K_{t_0}H$. Then we have

$$\bigotimes_{i=1}^{n} F_i = \left(\bigotimes_{i=1}^{n} P_i\right) \oplus X,$$

where X is the direct sum $P_1^{(\varepsilon_1)} \otimes P_2^{(\varepsilon_2)} \otimes \cdots \otimes P_n^{(\varepsilon_n)}$, $\varepsilon = 0,\ 1$, $P_j^{(0)} = P_j$, $P_j^{(1)} = P_j'$, and not all ε_i's are 0. Clearly this decomposition is invariant under the Γ action and therefore $\bigotimes_{i=1}^{n} P_i$ is a direct summand of $\bigotimes_{i=1}^{n} F_i$ over $K_t\Gamma$. Hence it is sufficient to show that the $K_t\Gamma$ module $B = \bigotimes_{i=1}^{n} F_i$ is projective.

The next reduction consists of decomposing the module B into certain $K_t\Gamma$ submodules and showing their projectiveness. These simple submodules are the linear span of Γ orbits in a basis of B.

Let $(b_\alpha)_{\alpha \in J}$ be a $K_{t_0}H$-basis of F. Then $(hb_\alpha)_{h \in H, \alpha \in J}$ is a K-basis of F and clearly the set $W=\{h_1 b_{\alpha_1}\otimes\cdots\otimes h_n b_{\alpha_n} : h_i\in H,\ \alpha_i\in J\}$ is a K-basis of B. From the definition of the action of Γ on B we see that the union $W\cup(-W)$ is Γ-invariant and specifically, for each $w\in W$ and $g\in\Gamma$

$$gw=w' \qquad \text{or} \qquad gw=-w' \qquad \text{for some} \quad w'\in W.$$

This action induces a permutation on W by

$$\pi(g)w=\{\pm gw\}\cap W.$$

Let $W=\bigcup_r W_r$ be the decomposition of W into different orbits under this permutation and let V_r be the K-free module generated by the orbit W_r. Then $B\cong\sum_r V_r$ as K-free modules. Moreover, since the set $W_r\cup(-W_r)$ is Γ-invariant, $B\cong\sum_r V_r$ is an isomorphism of $K_t\Gamma$ modules. Thus it suffices to show that the $K_t\Gamma$ module V_r is projective. In order to do that, let us describe the structure of this module in a different way.

Fix an r and an element $w_r=h_1 b_1\otimes\cdots\otimes h_n b_n$ in W_r. Denote by Γ_r the stabilizer of w_r under the permutation π, i.e.,

$$\Gamma_r=\{g\in\Gamma : gw_r=\pm w_r\}.$$

Let $X_{rs}=\{g_{rs}\}_{s\in I}$ be a set of representatives for the cosets of Γ_r in Γ. Evidently $W_r=\{\pi(g_{rs})\,w_r\}$ and $K_t\Gamma\cdot w_r=V_r$; i.e., there exists an epimorphism of $K_t\Gamma$ modules

$$f_r : K_t\Gamma\to V_r$$
$$z\mapsto z\cdot w_r.$$

Write z as $\sum_s g_{rs}a_s$ where $a_s\in K_t\Gamma_r$. Then $a_s w_r=y_s w_r$, $y_s\in K$ and hence

$$f_r(z)=\sum_s g_{rs}a_s w_r=\sum_s (\pm g_{rs}(y_s))\,\pi(g_{rs})\,w_r.$$

Since all the $\pi(g_{rs})\cdot w_r$'s are different elements of the basis W_r, we conclude that $f_r(z)=0$ if and only if all the $g_{rs}\cdot y_s$'s are zero. Equivalently the y_s's are zero, i.e., $a_s w_r=0$ for all s. Let I_r be the annihilator of w_r in $K_t\Gamma_r$, i.e.,

$$I_r=\{\eta\in K_t\Gamma_r \mid \eta w_r=0\}$$

then I_r is a left ideal in $K_t\Gamma_r$. By the above discussion, it is clear that $\mathrm{Ker}(f_r)=K_t\Gamma\cdot I_r$ and $V_r\cong K_t\Gamma/(K_t\Gamma\cdot I_r)$. Thus, if we show that I_r is a direct summand of $K_t\Gamma_r$ (as a $K_t\Gamma_r$ module) then, obviously $K_t\Gamma\cdot I_r$ and therefore also $V_r=K_t\Gamma/(K_t\Gamma\cdot I_r)$ are direct summands of $K_t\Gamma$ (over $K_t\Gamma$), proving the projectiveness of V_r.

Before we proceed to prove that I_r is a direct summand of $K_t\Gamma_r$ we wish to show that Γ_r is a finite group. Indeed let $h \in H \cap \Gamma_r$ and let $\sigma_1, ..., \sigma_n$ be representatives for the cosets of H in Γ. Assuming $\sigma_1 = 1$ we have $h^{-1}\sigma_1 = \sigma_1 h^{-1}$ (as a particular case of $g^{-1}\sigma_i = \sigma_{v_i} h_{v_i}^{-1}$). Then by definition of the action of Γ on B we have

$$hw_r = h(h_1 b_1 \otimes \cdots \otimes h_n b_n) = \pm hh_1 b_1 \otimes \cdots \otimes \cdots$$

and since $h \in \Gamma_r$, we known that $hw_r = \pm w_r$. Hence $hh_1 b_1 = h_1 b_1$, i.e., $h = 1$. This shows that $H \cap \Gamma_r = \{1\}$. Using the assumption that H is of finite index in Γ we have

$$|\Gamma_r| = |\Gamma_r : H \cap \Gamma_r| \leqslant |\Gamma : H| < \infty.$$

Now, we are ready to prove that the left ideal I_r is a direct summand of $K_t\Gamma_r$. As mentioned before, for $g \in \Gamma_r$, $gw_r = w_r$ or $gw_r = -w_r$. This defines a map sgn.: $\Gamma_r \to \mathbb{Z}_2 = \{-1, 1\}$ which, evidently is an homomorphism of groups. We will introduce a *new* action of $K_t\Gamma_r$ on K. For $xg \in K_t\Gamma_r$ and $y \in K$, define

$$xg[y] = \operatorname{sgn}(g)\, xg(y),$$

where $g(y)$ is the old action of g on $y \in K$.

A simple computation shows that this is a well defined action. Denote this $K_t\Gamma_r$ module by $\bar{K}$. We claim that there exists a short exact sequence of $K_t\Gamma_r$-modules

$$0 \to I_r \xrightarrow{i} K_t\Gamma_r \xrightarrow{\varphi} \bar{K} \to 0, \tag{2.2.10}$$

where i is the natural inclusion, and φ is given by

$$\varphi : K_t\Gamma_r \to \bar{K}$$
$$\sum_i x_i g_i \mapsto \sum_i \operatorname{sgn}(g_i)\, x_i.$$

Obviously, φ is K-linear and onto. Let us show that φ is a $K_t\Gamma_r$ morphism, i.e., $\varphi g(\sum x_i g_i) = g \cdot [\varphi(\sum x_i g_i)]$ for each $g \in \Gamma_r$, $\sum x_i g_i \in K_t\Gamma_r$. Indeed

$$\begin{aligned}\varphi\left(g \sum x_i g_i\right) &= \varphi\left(\sum_i g(x_i)\, gg_i\right) = \sum_i \operatorname{sgn}(gg_i)\, g(x_i) \\ &= \operatorname{sgn}(g) \sum_i \operatorname{sgn}(g_i)\, g(x_i) = g\left[\sum_i \operatorname{sgn}(g_i)\, x_i\right] \\ &= g \cdot \left[\varphi\left(\sum_i x_i g_i\right)\right].\end{aligned}$$

Finally, we show the exactness of the sequence in the term $K_t\Gamma_r$ of 2.2.10. $\sum_i x_i g_i$ in I_r means that $(\sum x_i g_i)\, w_r = 0$ or $(\sum \operatorname{sgn}(g_i)\, x_i)\, w_r = 0$. However, w_r is a basis element over K, so $\sum \operatorname{sgn}(g_i)\, x_i = 0$, i.e., $\sum x_i g_i \in \operatorname{Ker}(\varphi)$. From the short exact sequence, we see that to prove that I_r is a $K_t\Gamma_r$ direct summand of $K_t\Gamma_r$ is equivalent to showing that the morphism $\varphi\colon K_t\Gamma_r \to \bar{K}$ splits. By assumption there exists an identity decomposition of Γ_r; i.e., there exists $x_r \in K$, s.t.

$$\sum_{g_i \in \Gamma_r} g_i(x_r) = 1.$$

Define

$$\begin{aligned} &\psi\colon \bar{K} \to K_t\Gamma_r \\ &\qquad x \mapsto x \sum_i \operatorname{sgn}(g_i)\, g_i x_r = \sum_i \operatorname{sgn}(g_i)\, x g_i(x_r)\, g_i. \end{aligned}$$

We must show that ψ is a $K_t\Gamma_r$ morphism and that $\varphi\psi = id_{\bar{K}}$.

We start by showing that $\psi(g[x]) = g(\psi(x))$, $g \in \Gamma$, $x \in K$.

$$\begin{aligned} \psi(g[x]) &= \psi(\operatorname{sgn}(g)\; g(x)) \\ &= \operatorname{sgn}(g)\; g(x) \sum_j \operatorname{sgn}(g_j)\, g_j x_r = g(x) \sum_j \operatorname{sgn}(gg_j)\, g_j x_r \end{aligned}$$

while

$$g(\psi(x)) = gx \sum_i \operatorname{sgn}(g_i)\, g_i x_r = g(x) \sum_i \operatorname{sgn}(g_i)\, gg_i x_r.$$

Changing (say in the last sum) the summation index by $gg_i = g_j$ and then $\operatorname{sgn}(g_i) = \operatorname{sgn}(g^{-1}g_j) = \operatorname{sgn}(gg_j)$ we get the desired equality.

Finally we check that $\varphi\psi = id_{\bar{K}}$

$$\begin{aligned} \varphi\left(\sum_i \operatorname{sgn}(g_i)\, g_i x_r\right) &= \varphi\left(\sum_i g_i(x_r) \operatorname{sgn}(g_i)\; g_i\right) \\ &= \sum_i g_i(x_r) \operatorname{sgn}^2(g_i) = \sum_i g_i(x_r) = 1. \end{aligned}$$

This completes the proof of the proposition.

Now the whole theorem follows from the following lemma (an analogue to [Sw, Theorem 9.1]).

2.2.11. LEMMA. *Let Γ be a group acting on a commutative ring K and let H be a subgroup of Γ of finite index then either*

$$\operatorname{proj\,dim}_{K_t\Gamma} K = \operatorname{proj\,dim}_{K_{t_0}H} K$$

or

$$\text{proj dim}_{K_t\Gamma} K = \infty.$$

The proof is omitted as it is similar to the proof in [Sw].

3. Application of Duality to Homological Dimensions of Crossed Products

In order to compute homological dimensions associated with the crossed product $K_t^\alpha\Gamma$, using Poincaré duality of $K_t\Gamma$, we need to relate the homology groups of these algebras.

3.1.1. Lemma. *Let K be a field and let A, B be $K_t^\alpha\Gamma$ modules, then there exist isomorphisms*

$$\operatorname{Tor}_i^{K_t^\alpha\Gamma}(A, B) \cong \operatorname{Tor}_i^{K_t\Gamma}(A \otimes_K B, K) \tag{3.1.2}$$

$$\operatorname{Ext}^i_{K_t^\alpha\Gamma}(A, B) \cong \operatorname{Ext}^i_{K_t\Gamma}(K, \operatorname{Hom}_K(A, B)). \tag{3.1.3}$$

In the first isomorphism A and B are right and left modules, respectively, while in the second both are left modules.

Proof. In the first isomorphism $A \otimes_K B$ has a $K_t\Gamma$ right diagonal structure

$$(a \otimes b)\, x\sigma = axu_\sigma \otimes u_\sigma^{-1}b, \qquad x\sigma \in K_t\Gamma,\ u_\sigma \in K_t^\alpha\Gamma$$

(see 1.3).

Let M be a left $K_t\Gamma$ module. It is easy to show that the map

$$(A \otimes_K B) \otimes_{K_t\Gamma} M \to A \otimes_{K_t^\alpha G} (B \otimes_K M)$$
$$(a \otimes b) \otimes m \mapsto a \otimes (b \otimes m)$$

is a natural isomorphism. We show that this isomorphism induces the desired isomorphisms of the homology groups, using the Grothendieck's spectral sequence [Gr, Theorem 2.4.1]. Indeed, if P is a right projective $K_t^\alpha\Gamma$ module then $P \otimes_K B$ is a right projective $K_t\Gamma$ module. In particular $P \otimes_K B$ is $-\otimes_{K_t\Gamma} K$ acyclic. Thus, Grothendieck's theorem applies and since gl dim $K = 0$, we get isomorphisms

$$\operatorname{Tor}_*^{K_t\Gamma}(A \otimes_K B, K) \simeq \operatorname{Tor}_*^{K_t^\alpha\Gamma}(A, B).$$

Similarly, if A, B are left $K_t^{\alpha}\Gamma$ modules, and L is a left $K_t\Gamma$ module, then there exists a natural isomorphism

$$\operatorname{Hom}_{K_t\Gamma}(L, \operatorname{Hom}_K(A, B)) \simeq \operatorname{Hom}_{K_t^{\alpha}\Gamma}(L \otimes_K A, B)$$

which induces isomorphisms of the cohomology groups, namely

$$\operatorname{Ext}^*_{K_t\Gamma}(K, \operatorname{Hom}_K(A, B)) \simeq \operatorname{Ext}^*_{K_t^{\alpha}\Gamma}(A, B).$$

Here one shows that if B is an injective $K_t^{\alpha}\Gamma$-module then $\operatorname{Hom}_K(A, B)$ is injective over $K_t\Gamma$ and hence is $\operatorname{Hom}_{K_t\Gamma}(K, -)$ acyclic.

3.1.4. Corollary. gl dim $K_t^{\alpha}\Gamma \leqslant$ gl dim $K_t\Gamma =$ proj $\dim_{K_t\Gamma} K$.

Proof. The inequality follows immediately from the isomorphism 3.1.3 and since always proj $\dim_{K_t\Gamma} K \leqslant$ gl dim $K_t\Gamma$ the same isomorphism with $\alpha = 1$ shows the equality.

3.2. *2-Cocycles That Do Not Lower the (Weak) Global Dimension of Crossed Products*

In this section we prove Theorem 0.9 and some of its consequences.

First we consider the case of trivial action, i.e., $t = 0$ and $K\Gamma$ is a crossed product with orientable Poincaré duality.

Assume gl dim $K\Gamma = n$ ($=$proj $\dim_{K\Gamma} K$). If w.gl dim $K^{\alpha}\Gamma = n$ then there exist $K^{\alpha}\Gamma$ modules A and B s.t. $\operatorname{Tor}_n^{K^{\alpha}\Gamma}(A, B) \neq 0$.

By Lemma 3.1.1 we have $\operatorname{Tor}_n^{K\Gamma}(A \otimes_K B, K) \neq 0$. Using Poincaré duality (condition (2) in the definition of $\mathscr{A}$) we get $\operatorname{Ext}^0_{K\Gamma}(K, A \otimes_K B) \neq 0$, i.e., $\operatorname{Hom}_{K\Gamma}(K, A \otimes_K B) = (A \otimes_K B)^{\Gamma} \neq 0$ where X^{Γ} denotes the Γ-invariant elements in the $K\Gamma$ module X. The following proposition explains the implication of this situation in the $K^{\alpha}\Gamma$ modules A and B separately.

3.2.1. Proposition. *Let A and B be $K^{\alpha}\Gamma$ modules (right and left, respectively) and let $A \otimes_K B$ be the left $K\Gamma$ module, with the diagonal action, then $(A \otimes_K B)^{\Gamma} \neq 0$ if and only if there exist $K^{\alpha}\Gamma$ sub-modules V_A and V_B (of A and B, respectively) which are finite dimensional over K and if $V_B^* = \operatorname{Hom}_K(V_B, K)$ is the conjugate module of V_B with the left $K^{\alpha^{-1}}\Gamma$ structure defined in* 1.4, *then furnishing V_B^* with the right $K^{\alpha}\Gamma$ induced structure it becomes isomorphic to V_A.*

Proof. Let $0 \neq z = \sum_{i=1}^{s} a_i \otimes b_i \in (A \otimes_K B)^{\Gamma}$ and assume that z is of minimal length. This minimality implies that the elements $\{a_i\} \subset A$ (and similarly the elements $\{b_i\} \subset B$) which appear in z are linearly independent over K. Denote by V_A and V_B the subspaces spanned by the sets $\{a_i\}$ and $\{b_i\}$, respectively. Obviously $\dim_K V_A = \dim_K V_B < \infty$.

We are to show that the vector spaces V_A and V_B are $K^\alpha\Gamma$ submodules of A and B, respectively. Indeed choose bases $\{a_j\}$ for A and $\{b_j\}$ for B over K, which start with the bases of V_A and V_B.

Since $z \in (A \otimes_K B)^\Gamma$,

$$\sigma\left(\sum_{i=1}^{s} a_i \otimes b_i\right) = \sum_{i=1}^{s} a_i u_\sigma^{-1} \otimes u_\sigma b_i = \sum_{i=1}^{s} a_i \otimes b_i, \qquad \sigma \in \Gamma.$$

The element u_σ acts K-linearly on A and B therefore

$$\begin{aligned} a_i u_\sigma^{-1} &= a_1 p(i,1) + a_2 p(i,2) + \cdots + a_s p(i,s) + \cdots + a_m p(i,m) \\ u_\sigma b_i &= q(i,1)\, b_1 + q(i,2)\, b_2 + \cdots + q(i,s)\, b_s + \cdots + q(i,m)\, b_m, \end{aligned} \tag{3.2.2}$$

where $p(i,j), q(i,j) \in K$ and where $m \geqslant s$ is big enough to satisfy 3.2.2 for all $i = 1, ..., s$.

Replacing 3.2.2 in $\sum_{i=1}^{s} a_i u_\sigma^{-1} \otimes u_\sigma b_i$ yields

$$\begin{aligned} \sigma\left(\sum_{i=1}^{s} a_i \otimes b_i\right) &= \sum_{i=1}^{s} a_i u_\sigma^{-1} \otimes u_\sigma b_i = \sum_{i=1}^{s} \left(\sum_{j=1}^{m} a_j p(i,j)\right) \otimes_K \left(\sum_{r=1}^{m} q(i,r)\, b_r\right) \\ &= \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{r=1}^{m} a_j p(i,j)\, q(i,r) \otimes b_r \\ &= \sum_{j=1}^{m} \sum_{r=1}^{m} \sum_{i=1}^{s} a_j p(i,j)\, q(i,r) \otimes b_r \qquad \left(= \sum_{i=1}^{s} a_i \otimes b_i\right). \end{aligned}$$

Since the elements $\{a_j\}$ and $\{b_r\}$ are basis elements the last equality implies a set of equalities, one for each (j, r).

For (j, r) s.t. $1 \leqslant j = r \leqslant s$ we have $\sum_{i=1}^{s} p(i,j)\, q(i,j) = 1$ and for $j \neq r$ or $j = r > s$ it yields $\sum_{i=1}^{s} p(i,j)\, q(i,j) = 0$.

We claim that $p(i,j) = 0$ for $j > s$ and that $q(i,r) = 0$ for $r > s$. It is convenient to express these equations using matrix notation $P^T Q^T = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (P^T the transpose of P)

$$\begin{pmatrix} p(1,1) \cdots p(s,1) \\ \\ p(1,s) \cdots p(s,s) \\ p(1,m) \cdots p(s,m) \end{pmatrix}_{m \times s} \begin{pmatrix} q(1,1) & q(1,m) \\ \\ q(s,1) & q(s,m) \end{pmatrix}_{s \times m} = \begin{pmatrix} 1 & & & & \\ & 1 & 0 & & \\ & & \ddots & & 0 \\ & 0 & & & \\ & & & 1_{\,s \times s} & \\ & & 0 & & 0 \end{pmatrix}.$$

By this equality it is easily seen that the matrix $L_B = \left(\begin{smallmatrix} q(1,1) \cdots q(1,s) \\ q(s,1) \cdots q(s,s) \end{smallmatrix}\right)$ is invertible and its inverse is $L_A = \left(\begin{smallmatrix} p(1,1) \cdots p(s,1) \\ p(1,s) \cdots p(s,s) \end{smallmatrix}\right)$.

Moreover, for the lth row $\bar{p}_l$ $(l > s)$ of the matrix P^T we have $\bar{p}_l L_B = 0$. L_B is invertible, hence $\bar{p}_l = 0$, i.e., $p(i,j) = 0$ for $j > s$, $i = 1, ..., s$.

Similarly one can show that the column vector $\bar{q}_l$ in Q^T $(l>s)$ is the zero vector, i.e., $q(i, j)=0$, $j>s$, $i=1, ..., s$ as desired. If we denote by P_σ and Q_σ the restrictions of P and Q to V_A and V_B, respectively, then by the above discussion we have $P_\sigma = Q_\sigma^{-1}$.

Let $V_B^* = \operatorname{Hom}_K(V_B, K)$ be the left $K^{\alpha^{-1}}\Gamma$ module defined by $w_\sigma h = h \circ u_\sigma^{-1}$, $w_\sigma \in K^{\alpha^{-1}}\Gamma$, $u_\sigma \in K^\alpha\Gamma$. Hence V_B^* possesses a right $K^\alpha\Gamma$ structure

$$(hu_\sigma)(x) = (w_\sigma^{-1}h)(x) = h(u_\sigma x), \qquad u_\sigma \in K^\alpha\Gamma,\ w_\sigma \in K^{\alpha^{-1}}\Gamma.$$

With this structure

$$\eta\colon V_A \to \operatorname{Hom}_K(V_B K)$$
$$a_i \mapsto h_i, \qquad h_i(b_j) = \delta_{ij}$$

is a $K^\alpha\Gamma$ isomorphism. Obviously this map is $1-1$ onto and K-linear. We have to show that η commutes with the right actions of $K^\alpha\Gamma$. Since $\{b_j\}$ is a basis of V_B, it sufficies to show that $\eta(a_i u_\sigma)(b_j) = (\eta(a_i)u_\sigma)(b_j)$.

The right hand side yields

$$(\eta(a_i)u_\sigma)(b_j) = \eta(a_i)(u_\sigma b_j) = \eta(a_i)\left(\sum_{r=1}^{s} q(j, r)\, b_r\right)$$
$$= h_i\left(\sum_{r=1}^{s} q(j, r)\, b_r\right) = q(j, i)$$

while the left hand side yields

$$\eta(a_i u_\sigma)(b_j) = \eta\left(\sum a_r q(r, i)\right)(b_j) = \sum_r (h_r q(r, i))\, b_j = q(j, i)$$

as desired.

To prove the converse assume $V_A \subset A$, $V_B \subset B$ are K-finite dimensional $K^\alpha\Gamma$ submodules s.t. $\varphi\colon V_A \cong \operatorname{Hom}_K(V_B, K)$ is an isomorphism of right $K^\alpha\Gamma$ modules. Let $\{a_i\}$ be a basis for V_A and $\{b_i\}$ the dual basis of the image $\{\varphi(a_i)\}$. The element, $\Delta = \sum_{i=1}^{s} a_i \otimes b_i$ is a Γ-invariant element of $A \otimes_K B$, i.e., $(A \otimes_K B)^\Gamma \neq 0$ as desired. This completes the proof of the proposition.

The combination of this result and the discussion before the proposition proves the following theorem.

3.2.3. THEOREM. *Let $K^\alpha\Gamma$ be a crossed product which belongs to the family $\mathscr{A}$. Let* gl dim $K\Gamma = n$. *Then*

(1) w.gl dim $K^\alpha\Gamma = n$ *if and only if there exists a non-trivial unitary $K^\alpha\Gamma$ module of finite dimension over K.*

(2) *flat* $\dim_{K^\alpha\Gamma}(A)=n$ *if and only if* A *has a* $K^\alpha\Gamma$*-submodule which is of finite dimension over* K (A *is a* $K^\alpha\Gamma$ *module*).

(3) $\mathrm{Tor}_n^{K^\alpha\Gamma}(A, B)\neq 0$ *if and only if* A *and* B *contains* $K^\alpha\Gamma$*-submodules* V_A *and* V_B, *respectively, s.t.* $\dim_K V_A=\dim_K V_B<\infty$ *and* $V_A\cong \mathrm{Hom}_K(V_B K)$ *as right* $K^\alpha\Gamma$ *modules.*

Extension to the case where the action t is not trivial. Let $K_t^\alpha\Gamma$ be a crossed product in $\mathscr{A}$ where the trivial crossed $K_t\Gamma$ has orientable Poincaré duality of dimension n. If w.gl dim $K_t^\alpha\Gamma=$ gl dim $K_t\Gamma=n$ then there exists $K_t^\alpha\Gamma$ modules M, N such that $\mathrm{Tor}_n^{K_t^\alpha\Gamma}(M, N)\neq 0$. By Lemma 3.1.1, $\mathrm{Tor}_n^{K_t\Gamma}(M\otimes_K N, K)\neq 0$. Using the duality of $K_t\Gamma$ we get as in the the preceding discussion to Proposition 3.2.1,

$$\mathrm{Ext}^0_{K_t\Gamma}(K, M\otimes_K N)\simeq \mathrm{Hom}_{K_t\Gamma}(K, M\otimes_K N)\simeq (M\otimes_K N)^\Gamma\neq 0.$$

Let $0\neq\sum m_i\otimes n_i\in(M\otimes N)^\Gamma$ be of minimal length, then as in Proposition 3.2.1 the set $\{m_i\}$ in M (and similarly the set $\{n_i\}$ in N) is linearly independent over K and in particular over $K^\Gamma=k$. As in the same proposition, the spaces spanned by $\{m_i\}$ and $\{n_i\}$ over k, are Γ-invariants and of finite dimension over k. Therefore the vector spaces generated by $\{m_i\}$ and $\{n_i\}$ over K are $K_t^\alpha\Gamma$ modules and clearly of finite dimension over K.

Finally, we wish to extend the above result to the non-orientable case.

Assume that $K_t\Gamma$ possesses non-orientable Poincaré duality. Denote by $\bar{K}$ the dualizing module. By Definition 0.1 the module $\bar{K}\otimes_K\bar{K}$ with the right diagonal action is isomorphic to the $K_t\Gamma$ principal module (with the right induced action). If w.gl dim $K_t^\alpha\Gamma=$ gl dim $K_t\Gamma=n$ then again, there exist $K_t^\alpha\Gamma$ modules such that $\mathrm{Tor}_n^{K_t^\alpha\Gamma}(M, N)\neq 0$ and by Lemma 3.1.1, $\mathrm{Tor}_n^{K_t\Gamma}(M\otimes_K N, K)\neq 0$. Since $\bar{K}\otimes_K\bar{K}\simeq K$ we get $\mathrm{Tor}_n^{K_t\Gamma}((\bar{K}\otimes_K\bar{K})\otimes_K(M\otimes_K N), K)\neq 0$ and by duality

$$\begin{aligned}\mathrm{Ext}^0_{K_t\Gamma}(K, \bar{K}\otimes_K M\otimes_K N)&\cong \mathrm{Hom}_{K_t\Gamma}(K, \bar{K}\otimes_K M\otimes_K N)\\&\cong(\bar{K}\otimes_K M\otimes_K N)^\Gamma\neq 0.\end{aligned}$$

As before this implies the existence of non-trivial unitary $K_t^\alpha\Gamma$ modules of finite dimension over K, which satisfy (0.10).

EXAMPLE. Assume an action of Γ on K and $\alpha\in H^2(\Gamma, K^*)$. Let H be a subgroup of Γ of finite index such that $\alpha\mid H=1$, i.e., $\mathrm{res}(\alpha)=1$ where

$$\mathrm{res}\colon H^2(\Gamma, K^*)\to H^2(H, K^*).$$

The algebra K_tH is naturally embedded in $K_t^\alpha\Gamma$ and assumming w.gl dim $K_t\Gamma<\infty$ we have

$$\text{w.gl dim } K_tH\leqslant \text{w.gl dim } K_t^\alpha\Gamma\leqslant \text{w.gl dim } K_t\Gamma=\text{w.gl dim } K_tH.$$

Thus, for $K_t^\alpha\Gamma\in\mathscr{A}$, there exists a $K_t^\alpha\Gamma$ module of finite dimension over K. However, such a module exists for the general case by the induction $K_t^\alpha\Gamma\otimes_{K_tH}K$ where K is the principal module over K_tH.

As a result of Theorem 3.2.3 (and its extensions) we have

3.2.4. COROLLARY.

$$N=\{\alpha\in H^2(\Gamma,K^*):K_t^\alpha\Gamma\in\mathscr{A},\ \text{w.gl dim}\,K_t^\alpha\Gamma=\text{gl dim}\,K_t\Gamma\}$$

is a subgroup of $H^2(\Gamma,K^*)$.

Proof. Let $\text{gl dim}\,K_t\Gamma=n$. If $\text{w.gl dim}\,K_t^\alpha\Gamma=n$ then by Theorem 3.2.3 and extensions there exists a left $K_t^\alpha\Gamma$ module V of finite dimension over K. The dual module $\text{Hom}_K(V,K)$ is a left $K_t^{\alpha^{-1}}\Gamma$ module by $(z_\sigma h)(v)=h(u_\sigma^{-1}u)$, $z_\sigma\in K_t^{\alpha^{-1}}\Gamma$, $u_\sigma\in K_t^\alpha\Gamma$, $h\in\text{Hom}_K(V,K)$.

Since it is of finite dimension over K, $\text{gl dim}\,K_t^{\alpha^{-1}}\Gamma=n$. If ${}_\alpha V$ and ${}_\beta V$ are left $K_t^\alpha\Gamma$ and $K_t^\beta\Gamma$ modules of finite dimension over K then clearly ${}_\alpha V\otimes_K{}_\beta V$ is a $K_t^{\alpha\beta}\Gamma$ module with the diagonal action. $\dim{}_\alpha V\otimes_K{}_\beta V=\dim{}_\alpha V\cdot\dim{}_\beta V<\infty$ hence $\text{w.gl dim}\,K_t^{\alpha\beta}\Gamma=n$. The identity of $H^2(\Gamma,K^*)$ belongs to N since $\text{w.gl dim}\,K_t\Gamma=\text{gl dim}\,K_t\Gamma$.

For members of the family $\mathscr{A}$, we discussed 2-cocycles which yield (in $K_t^\alpha\Gamma$) the maximum weak global dimension, i.e., elements $\alpha\in H^2(\Gamma,K^*)$ which do not lower the weak global dimension and in some sense the "twisting" made by them is trivial.

A first step in which we showed this triviality was made by proving that $\alpha,\ \beta\in H^2(\Gamma,K^*)$ s.t. $\text{w.gl dim}\,K_t^\alpha\Gamma=\text{w.gl dim}\,K_t^\beta\Gamma=\text{gl dim}\,K_t\Gamma=n$ then also $\text{w.gl dim}\,K_t^{\alpha\beta}\Gamma=n$. We make an additional step by

3.2.5. PROPOSITION. *If* $\alpha,\ \beta\in H^2(\Gamma,K^*)$ *such that*

(1) $\text{w.gl dim}\,K_t^\alpha\Gamma=\text{gl dim}\,K_t\Gamma=n$

(2) $\text{w.gl dim}\,K_t^\beta\Gamma=r\leqslant n$

then $\text{w.gl dim}\,K_t^{\alpha\beta}\Gamma=r$. *(We still assume that the crossed products belong to* $\mathscr{A}$.*)*

Proof. Let $\text{w.gl dim}\,K_t^{\alpha\beta}\Gamma=s$. By the long exact sequence in homology, we have that the functor $\text{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta},\ \ )$ is left exact and that there exist right and left $K_t^{\alpha\beta}\Gamma$ modules $M_{\alpha\beta},{}_{\alpha\beta}N$ such that

$$\text{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta},{}_{\alpha\beta}N)\neq 0. \tag{3.2.6}$$

Consider the natural morphism

$$ {}_{\alpha\beta}N\to\text{Hom}_K({}_\alpha M,{}_{\alpha\beta}N\otimes{}_\alpha M)$$

$$n\mapsto\eta_n,\qquad \eta_n(m)=n\otimes m,$$

where ${}_{\alpha}M \neq 0$ is a left $K_t^{\alpha}\Gamma$ module. It is straightforward to show that this is a monomorphism of $K_t^{\alpha\beta}\Gamma$-modules. (For the $K_t^{\alpha\beta}\Gamma$ structure of $\mathrm{Hom}_K({}_{\alpha}M, {}_{\alpha\beta}N \otimes_K {}_{\alpha}M)$ see Sections 1, 1.2, and 1.4.) Since w.gl dim $K_t^{\alpha}\Gamma = n$, we can assume that ${}_{\alpha}M$ is of finite dimension over K. By left exactness of $\mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, -)$ we get exactness of

$$0 \to \mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, {}_{\alpha\beta}N) \to \mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, \mathrm{Hom}_K({}_{\alpha}M, {}_{\alpha\beta}N \otimes_K {}_{\alpha}M)).$$

Since $\mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, {}_{\alpha\beta}N) \neq 0$ then also

$$\mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, \mathrm{Hom}_K({}_{\alpha}M, {}_{\alpha\beta}N \otimes_K {}_{\alpha}M)) \neq 0.$$

The module ${}_{\alpha}M$ is of finite dimension over K therefore there is a natural isomorphism of $K_t^{\alpha\beta}\Gamma$ modules

$$\mathrm{Hom}_K({}_{\alpha}M, {}_{\alpha\beta}N \otimes_K {}_{\alpha}M) \cong {}_{\alpha^{-1}}M^* \otimes_K {}_{\alpha\beta}N \otimes_K {}_{\alpha}M, \tag{3.2.7}$$

where ${}_{\alpha^{-1}}M^*$ is the dual module of ${}_{\alpha}M$ viewed as a $K_t^{\alpha^{-1}}\Gamma$ module. (The $K_t^{\alpha\beta}\Gamma$ structure of both sides of 3.2.7 are defined in Section 1.) Thus

$$\mathrm{Tor}_s^{K_t^{\alpha\beta}\Gamma}(M_{\alpha\beta}, {}_{\alpha^{-1}}M^* \otimes {}_{\alpha\beta}N \otimes {}_{\alpha}M) \neq 0.$$

However, this is natural isomorphic to $\mathrm{Tor}_s^{K_t^{\beta}\Gamma}(M_{\alpha\beta} \otimes M_{\alpha^{-1}}, {}_{\alpha^{-1}}M^* \otimes {}_{\alpha\beta}N)$ $(\neq 0)$ (replace the module K by $M_{\alpha\beta}$ in Lemma 3.1.1) showing that $r = $ w.gl dim $K_t^{\beta}\Gamma \geqslant s$. To prove the converse recall that w.gl dim $K_t^{\alpha^{-1}}\Gamma = n$ (because w.gl dim $K_t^{\alpha}\Gamma = n$ and Corollary 3.2.4). This implies

$$\text{w.gl dim } K_t^{\beta}\Gamma \geqslant \text{w.gl dim } K_t^{\alpha\beta}\Gamma \geqslant \text{w.gl dim } K_t^{\alpha^{-1}\alpha\beta}\Gamma = \text{w.gl dim } K_t^{\beta}\Gamma$$

as desired.

3.2.8. *Remark.* If in addition to the assumption that the crossed product belongs to $\mathscr{A}$, we assume that the global dimension can be computed by means of the functor Tor (e.g., G poly{cyclic or finite}) we get the same result estimating the global dimension of the crossed product instead of the weak global dimension.

One possible way for constructing 2-cocycles which lower the weak global dimension of a group ring $K\Gamma$ is the following.

Let $K^{\alpha}\Gamma$ be a crossed product in $\mathscr{A}$ such that w.gl dim $K^{\alpha}\Gamma = $ gl dim $K\Gamma$. Let V be a $K^{\alpha}\Gamma$ module of finite dimension over K. The action of Γ on K is trivial hence there exists a K-linear morphism of algebras

$$K^{\alpha}\Gamma \to \mathrm{End}_K(V).$$

Since $\mathrm{End}_K(V)$ is of finite dimension over K there is an ideal I in $K^{\alpha}\Gamma$ of finite co-dimension over K. Obviously the converse is also true. From this discussion it follows that if $K^\alpha\Gamma$ is a simple algebra and Γ is infinite then w.gl dim $K^\alpha\Gamma <$ gl dim $K\Gamma$.

Such crossed products are constructed for the free abelian group on n generators in [Ro, Lemma 4.2]. This shows that gl dim $K^\alpha\Gamma \leqslant n-1$ for Γ free abelian of rank n. For $n=2$, our considerations give another proof of Shamsuddin's result. It should be noticed that the proof that gl dim $K^\alpha\Gamma = 1$ for these examples ($n>2$) is much more difficult.

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References

[AR1] E. Aljadeff and S. Rosset, Global dimensions of crossed products, *J. Pure Appl. Algebra* **40** (1986), 103–113.

[AR2] E. Aljadeff and S. Rosset, Crossed products, cohomology, and equivariant projective representations, *J. Algebra* **121** (1989), 99–116.

[BE] R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, *Invent. Math.* **20** (1973), 103–124.

[Gr] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957), 119–221.

[MR] J. C. McConnell and J. C. Robson, "Non Commutative Noetherian Rings," Wiley, New York, 1987.

[Pa] D. A. Passman, "The Algebraic Structure of Group Rings." Wiley–Interscience, New York, 1977.

[Ro] S. Rosset, An invariant of free central extensions, *J. Pure Appl. Algebra* **48** (1987), 301–315.

[Se] J. P. Serre, "Local Fields," Springer-Verlag, New York/Berlin, 1979.

[Sh] A. Shamsuddin, A class of simple Noetherian domains, *J. London Math. Soc.* (2) **15** (1977), 213–216.

[Sw] G. Swan, Groups of cohomological dimension one, *J. Algebra* **12** (1969), 585–601.